

# On the Inversion of the Linearized Collision Operator

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Some features are discussed in connection with the representation of the linearized Boltzmann collision operator and its inversion. It is shown that under certain assumptions the inverse operator can be given explicitly as an integral kernel function.

## Introduction

A vast variety of problems in the kinetic theory of gases and gas mixtures has been treated successfully by means of moment methods, first in their original form introduced by Grad [1–3] and later in a generalized version based upon group-theoretical aspects, using spherical harmonics as irreducible representations of the group of three-dimensional rotations [4–6]. Especially in the theory of drift tubes considerable success has been achieved [7–10]. Within a theory for moderately weak fields an iteration scheme has been formulated previously [11] yielding analytic expressions for the linear transport coefficients as well as their lowest-order nonlinear extensions. The central part in the evaluations of such results is the inversion of the matrix of the linearized collision operator. This problem would be facilitated considerably if some progress could be made in an analytic treatment of the inversion procedure.

While there is a large number of treatments concerning the mathematical structure of the (linearized) collision operator and in particular its spectral properties (e.g. cf. Refs. [12–20]) the direct approach to the inverse operator seems to be rather unpopular. It is therefore the purpose of the present paper to discuss some aspects of this inversion and to demonstrate that the inverse operator can be given explicitly for a special example.

We start in Sect. I with a brief discussion of the connection between the collision elements as can be used in moment methods [20, 21] and a Fredholm-type integral operator representing the linearized Boltzmann collision operator. In Sect. II the results are specialized successively to elastic

collisions, to inverse power potentials, and to the rigid-sphere case. Based upon this discussion we proceed in Sect. III to establishing a differential equation for the isotropic part of the inverse collision kernel as a special example. Finally, the solution of this differential equation is given in Section IV.

## I. Representation of the Linearized Collision Operator as a Fredholm Operator

In our considerations we start with the widely discussed representation of the collision operator for isotropic elastic interactions in the Burnett basis system [20] (cf. Appendix). Let the velocity distribution functions  $f(i)$  for the particles of species  $i$  of a gaseous mixture be expanded as

$$f(i) = A_i^{n,l,m} \Phi_{n,l,m}(i). \quad (1.1)$$

(Note that we use the convention to sum over indices appearing once and only once as superscript and subscript.) Then the nonlinear collision operator can be expressed as

$$B(i, j)[f(i), f(j)] = \Phi_{n',l',m'}(i) \cdot B_{n,l,m;N,L,M}^{n',l',m'}(i, j) A_i^{n,l,m} A_j^{N,L,M}. \quad (1.2)$$

The conventional way to proceed is to use the “collision elements”

$$B_{n,l,m;N,L,M}^{n',l',m'}(i, j) = \sum_{l''} T_{q,l'',m'';Q,L',M'}^{n',l',m';0,0,0} V_{q'}^q(l'') \cdot T_{n,l,m;N,L,M}^{q',l'',m'';Q,L',M'} \quad (1.3)$$

of Eq. (1.2) for further evaluations [6, 20, 21]. The matrices  $T$  are the transformation matrices between products of Burnett basis functions  $\{\Phi_{n,l,m}(i) \Phi_{N,L,M}(j)\}$  and products of corresponding functions depending on the relative velocity

$$\mathbf{g} = \mathbf{c}_j - \mathbf{c}_i \quad (1.4)$$

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and the center-of-mass velocity

$$\mathbf{G} = M_j \mathbf{c}_j + M_i \mathbf{c}_i, \quad (1.5)$$

$\mathbf{c}_i$  and  $\mathbf{c}_j$  being the particle velocities and  $M_{i,j}$  the mass factors

$$M_{i,j} = m_{i,j}/(m_i + m_j). \quad (1.6)$$

Thus, the transformation matrices  $T$  are given as

$$T_{q,l'',m'';Q,L',M'}^{n',l',m';0,0,0} = \iint d^3g d^3G \Phi^{n',l',m'}(i) \cdot \Phi_{q,l'',m''}(\mathbf{g}) \Phi_{Q,L',M'}(\mathbf{G}) \quad (1.7)$$

and

$$T_{n,l,m;N,L,M}^{q',l',m';Q,L',M'} = \iint d^3c'_i d^3c'_j \Phi^{q',l',m'}(\mathbf{g}') \cdot \Phi_{Q,L',M'}(\mathbf{G}') \Phi_{n,l,m}(i') \Phi_{N,L,M}(j'). \quad (1.8)$$

The  $V$  matrices contain the physical information about the molecular interaction which in our case is assumed to be elastic:

$$V_q^{q'}(l'') = \int d\mathbf{g} g^3 \Phi_{l'',q'}^{q'}(\mathbf{g}) \Phi_{q,l''}(g) Q_{ij}^{(l'')}(g) \cdot \left( \frac{k_B T_i}{m_i} + \frac{k_B T_j}{m_j} \right)^{(q'-q-3)/2}. \quad (1.9)$$

The functions  $\Phi_{l'',q'}^{q'}(\mathbf{g})$  and  $\Phi_{q,l''}(g)$  are given in the Appendix,  $T_{i,j}$  are the temperatures of the particles of species  $i$  and  $j$ . The transport cross sections [22]

$$Q_{ij}^{(l'')}(g) = 2\pi \int_{-1}^{+1} d \cos \chi \sigma_{ij}(\chi, g) \cdot [1 - P_{l''}(\cos \chi)] \quad (1.10)$$

are suitably chosen linear combinations of standard expressions [23].

Instead of the matrices  $V$  we use the functions

$$V(\mathbf{g}, \mathbf{g}') \equiv \sum_{l''} \Phi_{q',l'',m''}^{q'}(\mathbf{g}') V_q^{q'}(l'') \cdot \Phi_{q',l'',m''}(\mathbf{g}) \quad (1.11)$$

and obtain from Eq. (1.2)

$$\begin{aligned} B(i, j) [f(i), f(j)] &= \Phi_{n',l',m'}(i) \iint d^3g' d^3g d^3G \Phi^{n',l',m'}(i) \\ &\cdot V(\mathbf{g}, \mathbf{g}') \Phi_{n,l,m}(i') \Phi_{N,L,M}(j') \\ &\cdot A_i^{n,l,m} A_j^{N,L,M}. \end{aligned} \quad (1.12)$$

Here we have used the completeness relation for the Burnett functions,

$$\Phi_{Q,L',M'}^{Q,L,M'}(\mathbf{G}') \Phi_{Q,L',M'}(\mathbf{G}) = \delta^3(\mathbf{G} - \mathbf{G}'). \quad (1.13)$$

Similarly we obtain from Eq. (1.12)

$$\begin{aligned} B(i, j) [f(i), f(j)] &= \iint d^3g' d^3g V(\mathbf{g}, \mathbf{g}') \\ &\cdot f_i(\mathbf{c}_i + M_j(\mathbf{g} - \mathbf{g}')) f_j(\mathbf{c}_i + M_j \mathbf{g} + M_i \mathbf{g}'). \end{aligned} \quad (1.14)$$

Here we have combined the functions  $\Phi_{n',l',m'}(i)$  with  $\Phi^{n',l',m'}(i)$  under the integral, and because of the corresponding  $\delta$  function we have been able to carry out the integration over  $\mathbf{G} = \mathbf{G}'$ . Furthermore we have substituted the generalized moments

$$A_{i,j}^{n,l,m} \equiv \int d^3c_{i,j} \Phi^{n,l,m}(i, j) f(i, j). \quad (1.15)$$

From Eq. (1.14) we can deduce two independent linear operators, substituting either  $f_i$  or  $f_j$  by the equilibrium distribution function

$$\begin{aligned} f_0(\mathbf{c}_{i,j}) &= (m_{i,j}/2\pi k_B T_{i,j})^{3/2} n_{i,j} \\ &\cdot \exp(-m_{i,j} c_{i,j}^2/2k_B T_{i,j}) \end{aligned} \quad (1.16)$$

with the particle densities  $n_i(\mathbf{x}, t)$  and  $n_j(\mathbf{x}, t)$ . Finally, shifting the integration over  $\mathbf{g}'$  to the particle velocities

$$\mathbf{c}'_{j,i} = \mathbf{c}_i + M_j \mathbf{g} \pm M_{i,j} \mathbf{g}', \quad (1.17)$$

we obtain

$$\begin{aligned} L_1(i, j) f(j) &\equiv \int d^3c'_j f_j(\mathbf{c}'_j) \\ &\cdot M_i^{-3} \int d^3g V(\mathbf{g}, (\mathbf{c}'_j - \mathbf{c}_i - M_j \mathbf{g})/M_i) \\ &\cdot f_{0i}((\mathbf{c}_i + M_j \mathbf{g} - M_j \mathbf{c}'_j)/M_i) \end{aligned} \quad (1.18)$$

and

$$\begin{aligned} L_2(i, j) f(i) &\equiv \int d^3c'_i f_i(\mathbf{c}'_i) \\ &\cdot M_j^{-3} \int d^3g V(\mathbf{g}, \mathbf{g} + (\mathbf{c}_i - \mathbf{c}'_i)/M_j) \\ &\cdot f_{0j}(\mathbf{g} + (\mathbf{c}_i - M_i \mathbf{c}'_i)/M_j). \end{aligned} \quad (1.19)$$

For a uniform gas we have to combine both results. Then, with  $M_i = M_j = 1/2$ , we have

$$\begin{aligned} Lf &\equiv 8 \int d^3c' f(\mathbf{c}') \int d^3g f_0(2\mathbf{c} - \mathbf{c}' + \mathbf{g}) \\ &\cdot \{V(\mathbf{g}, 2(\mathbf{c}' - \mathbf{c}) - \mathbf{g}) \\ &\quad + V(\mathbf{g}, 2(\mathbf{c} - \mathbf{c}') + \mathbf{g})\}. \end{aligned} \quad (1.20)$$

Thus we have achieved a representation of the linearized collision operators as Fredholm-type integral operators. In the next section we are going to give a brief discussion of the  $V$  functions. The actual discussion of the collision operator and its inverse, starting in Sect. III, will be concerned with the special case  $L_2$ , Equation (1.19). As the basic structure of all three operators (1.18)–(1.20) is very similar, there is no loss of generality in this special choice.

## II. The $V$ Functions for Elastic Collisions

We start the considerations of this section with the remark that the completeness relation (1.13) for the Burnett basis functions implies the fol-

lowing completeness relation for the “radial” parts (cf. Appendix):

$$\frac{2}{g} \delta(g^2 - g'^2) = \left( \frac{k_B T_i}{m_i} + \frac{k_B T_j}{m_j} \right)^{-3/2} \cdot \Phi_l^i(g') \Phi_{q,l}(g), \quad (2.1)$$

where there is no summation over  $l$  on the right-hand side. We apply the addition theorem for the spherical harmonics,

$$\sum_m Y_{l,m}^*(\hat{\mathbf{x}}) Y_{l,m}(\hat{\mathbf{y}}) = \frac{2l+1}{4\pi} P_l(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}), \quad (2.2)$$

$\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  being any two unit vectors, to the definition (1.11) of the  $V$  functions and obtain

$$\begin{aligned} V(\mathbf{g}, \mathbf{g}') &= \sum_l \frac{2l+1}{4\pi} P_l(\cos \Theta) \\ &\cdot 2 \int_0^\infty d\bar{g}^2 \delta(g^2 - \bar{g}^2) \delta(g'^2 - \bar{g}^2) Q_{ij}^{(l)}(\bar{g}) \\ &= \sum_l \frac{2l+1}{2\pi} P_l(\hat{\mathbf{g}} \cdot \hat{\mathbf{g}}') \delta(g^2 - g'^2) Q^{(l)}(g). \end{aligned} \quad (2.3)$$

For special cases the summation over  $l$  can be carried out: An expansion of the cross sections  $\sigma_{ij}(\cos \chi)$  in terms of Legendre polynomials yields

$$\sigma_{ij}(\cos \chi, g) \equiv \sum_l \sigma_{ij}^l(g) P_l(\cos \chi) \quad (2.4)$$

with

$$\sigma_{ij}^l(g) = \frac{2l+1}{2} \int d\cos \chi \sigma_{ij}(\cos \chi, g) P_l(\cos \chi). \quad (2.5)$$

Substituting this integral in the transport cross section (1.10), we have

$$Q_{ij}^{(l)}(g) = -\frac{4\pi}{2l+1} \sigma_{ij}^l(g) + 4\pi \sigma_{ij}^0(g). \quad (2.6)$$

In combination with Eq. (2.3) this yields

$$\begin{aligned} V(\mathbf{g}, \mathbf{g}') &= -2 \sigma_{ij}(\hat{\mathbf{g}} \cdot \hat{\mathbf{g}}, g) \delta(g^2 - g'^2) \\ &+ 2 \delta(g^2 - g'^2) \sigma_{ij}^0(g) \\ &\cdot \sum_l (2l+1) P_l(\hat{\mathbf{g}} \cdot \hat{\mathbf{g}}'). \end{aligned} \quad (2.7)$$

Applying the completeness relations (2.1) and (1.13), we can evaluate the second term on the right-hand side. We obtain

$$\begin{aligned} V(\mathbf{g}, \mathbf{g}') &= 4\pi g \sigma_{ij}^0(g) \delta^3(\mathbf{g} - \mathbf{g}') \\ &- 2 \sigma_{ij}(\hat{\mathbf{g}} \cdot \hat{\mathbf{g}}', g) \delta(g^2 - g'^2). \end{aligned} \quad (2.8)$$

For cut-off potentials, especially for rigid spheres, this formula can be used. For “soft” potentials,

however,  $\sigma_{ij}^0(g)$  as well as  $\sigma_{ij}$  diverge simultaneously; we have to use (2.3).

In the following sections we shall restrict ourselves to the extremely simple case of rigid spheres, where we have  $\sigma_{RS} = \sigma_{RS}^0$  and hence,

$$\begin{aligned} V(\mathbf{g}, \mathbf{g}') &= \sigma_{RS} (4\pi g \delta^3(\mathbf{g} - \mathbf{g}') \\ &- 2 \delta(g^2 - g'^2)). \end{aligned} \quad (2.9)$$

### III. The Inverse Collision Operator

In the previous sections we have discussed the collision operator

$$L f \equiv \int d^3 c' L(\mathbf{c}, \mathbf{c}') f(\mathbf{c}'). \quad (3.1)$$

We facilitate further considerations by restricting ourselves to the “one-temperature model”, i.e. both equilibrium distribution functions  $f_0(i)$  and  $f_0(j)$  contain the same temperature  $T_i = T_j = T$ . As a consequence of this restriction the well-known fact that  $L$  has, in correspondence to the existence of collision invariants, a certain set of linearly independent eigenfunctions belonging to the eigenvalue 0, implies

$$[P, L] = 0, \quad (3.2)$$

where  $P$  is the projector onto the subspace spanned by those eigenfunctions. In the case of our special choice  $L_2$ , Eq. (1.19), there is no degeneracy of the eigenvalue 0; the only invariant is the number of particles. Hence, the projector  $P$  acts on any function  $h$  of the space under consideration as

$$P h = \Phi_{0,0,0}(\mathbf{c}) \int d^3 c' h(\mathbf{c}'), \quad (3.3)$$

due to the choice of our basis functions such that  $\Phi_{0,0,0}$  is proportional to the equilibrium distribution function:

$$\begin{aligned} f_0(\mathbf{c}) &\equiv n(\mathbf{x}, t) \Phi_{0,0,0}(\mathbf{c}) \\ &= n(\mathbf{x}, t) (\beta/\pi)^{3/2} \exp(-\beta c^2), \end{aligned} \quad (3.4)$$

$\beta$  being the temperature factor

$$\beta_{i,j} = m_{i,j}/2k_B T. \quad (3.5)$$

Hence, we have to invert the operator  $(1 - P)L(1 - P)$ , i.e. the restriction of  $L$  to the subspace perpendicular to the one belonging to the eigenvalue 0. However, on this perpendicular subspace its analytical form is the same because of  $LP = PL = 0$ . The integral equation for the determination of the (restricted) inverse operator  $L^{-1}$  then

reads

$$\delta^3(\mathbf{c}' - \mathbf{c}) - \Phi_{0,0,0}(\mathbf{c}') \\ = \int d^3\mathbf{c}'' L^{-1}(\mathbf{c}', \mathbf{c}'') L(\mathbf{c}'', \mathbf{c}). \quad (3.6)$$

Consequently, the operator  $L^{-1}$  should not contain any contribution  $PL^{-1}$ .

In order to proceed in the discussion of Eq. (3.6), we expand the inverse operator  $L^{-1}$  as

$$L^{-1}(\mathbf{c}', \mathbf{c}'') \equiv \Phi_{n,l,m}(\mathbf{c}') \tilde{L}^{n,l,m}(\mathbf{c}'') \quad (3.7)$$

with  $(n, l) \neq (0, 0)$ . Then, in connection with the completeness relation (1.13) we obtain instead of Eq. (3.6)

$$\Phi^{n,l,m}(\mathbf{c}) = \int d^3\mathbf{c}' \tilde{L}^{n,l,m}(\mathbf{c}') L(\mathbf{c}', \mathbf{c}). \quad (3.8)$$

In our physically rather simple case with ab initio conservation of (relative) angular momentum the collision operator  $L$  as well as its inverse  $L^{-1}$  are scalar operators. Hence, we can write

$$\tilde{L}^{n,l,m}(\mathbf{c}') \equiv (2\beta_l)^{-n/2} \tilde{L}_l^n(\mathbf{c}') Y_{l,m}^*(\hat{\mathbf{c}}') \quad (3.9)$$

and assume that  $L(\mathbf{c}', \mathbf{c})$  depends on its arguments as  $L(\mathbf{c}', c; \cos \vartheta)$  with

$$\cos \vartheta = \hat{\mathbf{c}} \cdot \hat{\mathbf{c}}'. \quad (3.10)$$

In order to make use of these properties we multiply Eq. (3.8) by  $Y_{l,m}(\hat{\mathbf{c}})$ , sum up over  $m$ , and obtain

$$\Phi_l^n(\mathbf{c}) = 2\pi \int_0^\infty c'^2 dc' \tilde{L}_l^n(\mathbf{c}') \\ \cdot \int_{-1}^{+1} d \cos \vartheta P_l(\cos \vartheta) L(\mathbf{c}', c; \cos \vartheta). \quad (3.11)$$

After these preliminary remarks we proceed to more explicit considerations. We restrict ourselves to the case of rigid spheres as most obvious example for cut-off potentials allowing the decomposition of  $L$  into two parts,

$$L(\mathbf{c}', \mathbf{c}) = L^{(0)}(\mathbf{c}', \mathbf{c}) - L^{(1)}(\mathbf{c}', \mathbf{c}) \quad (3.12)$$

corresponding to Eq. (2.9), with the contributions (cf. Eq. (1.18))

$$L^{(0)}(\mathbf{c}', \mathbf{c}) \equiv 4\pi M^{-3} \sigma_{RS} n_j (\beta_j/\pi)^{3/2} \\ \cdot \int d^3g \delta^3(\mathbf{g} - (\mathbf{c} - \mathbf{c}')/M - \mathbf{g}) g \\ \cdot \exp \left[ -\beta_j \left( \mathbf{g} + \frac{1}{M} \mathbf{c}' - \frac{1-M}{M} \mathbf{c} \right)^2 \right] \quad (3.13)$$

and

$$L^{(1)}(\mathbf{c}', \mathbf{c}) \equiv 2M^{-3} \sigma_{RS} n_j (\beta_j/\pi)^{3/2} \\ \cdot \int d^3g \delta(g^2 - [\mathbf{g} + (\mathbf{c} - \mathbf{c}')/M]^2) \\ \cdot \exp \left[ -\beta_j \left( \mathbf{g} + \left( \frac{1}{M} \mathbf{c}' - \frac{1-M}{M} \mathbf{c} \right) \right)^2 \right], \quad (3.14)$$

the index  $j$  indicating that the temperature factor  $\beta$  and the particle density are those of the collision partner in equilibrium.

The easy part to discuss is the contribution  $L^{(0)}$ : We define a collision frequency

$$\nu_{RS} \equiv 4 \sigma_{RS} n_j (\pi/\beta_j)^{1/2}, \quad (3.15)$$

use the normalized kinetic energy

$$\varepsilon \equiv \beta_j c^2, \quad (3.16)$$

and obtain after some elementary manipulations

$$L^{(0)}(\mathbf{c}', \mathbf{c}) = \nu_{RS} \delta^3(\mathbf{c}' - \mathbf{c}) k(\varepsilon)/\sqrt{\varepsilon}, \quad (3.17)$$

abbreviating

$$k(\varepsilon) = (\varepsilon + 1/2) \gamma(\varepsilon, \frac{1}{2}) + \sqrt{\varepsilon} e^{-\varepsilon} \quad (3.18)$$

with the incomplete  $\Gamma$  function of the second argument  $1/2$ ,

$$\gamma(\varepsilon, \frac{1}{2}) = \int_0^\varepsilon e^{-\varepsilon'} / \sqrt{\varepsilon'} d\varepsilon'. \quad (3.19)$$

The starting point for the discussion of the more complicated term  $L^{(1)}$  shall be a shift of the integration over  $d^3g$  in Eq. (3.14) by  $\mathbf{c}'/M$ . For the integral over the solid angle of  $\mathbf{c}'$  in Eq. (3.11) we then have

$$2\pi \int_{-1}^{+1} d \cos \vartheta P_l(\cos \vartheta) \\ \cdot \int d^3g \delta([\mathbf{g} - \mathbf{c}'/M]^2 - [\mathbf{g} - \mathbf{c}/M]^2) \\ \cdot \exp \left\{ -\beta_j \left( \mathbf{g} - \frac{1-M}{M} \mathbf{c} \right)^2 \right\}. \quad (3.20)$$

We rewrite the integration over  $d \cos \vartheta$  as

$$2\pi \int_{-1}^{+1} d \cos \vartheta P_l(\cos \vartheta) \dots \\ = \frac{4\pi}{2l+1} \sum_m Y_{l,m}(\hat{\mathbf{c}}) \int d\Omega' Y_{l,m}^*(\hat{\mathbf{c}}') \dots \quad (3.21)$$

and change the order of the integrations over the solid angle  $d\Omega'$  and  $d^3g$ . Then, choosing the polar axis for the integration over  $d\Omega'$  along  $\mathbf{g}$ , the polar angle between  $\mathbf{g}$  and  $\mathbf{c}'$  being  $\vartheta'$ , we obtain for the integral over  $d\Omega'$

$$\left( \frac{4\pi}{2l+1} \right)^{1/2} 2\pi \sum_m Y_{l,m}(\hat{\mathbf{c}}) \delta_0^m \int_{-1}^{+1} d \cos \vartheta' P_l(\cos \vartheta') \\ \cdot \delta([c'^2 - c^2]/M^2 - 2g c' \cos \vartheta'/M + 2g c \cos \Theta/M) \quad (3.22)$$

with

$$\cos \Theta = \hat{\mathbf{g}} \cdot \hat{\mathbf{c}}. \quad (3.23)$$

Defining the function (3.24)

$$z(\cos \Theta) \equiv \frac{M}{2gc'} \left[ \frac{1}{M^2} (c'^2 - c^2) + \frac{2}{M} gc \cos \Theta \right],$$

we immediately obtain for the integral (3.22)

$$2\pi P_l(\cos \Theta) P_l(z(\cos \Theta)) M/2gc', \quad (3.25)$$

if  $|z| \leq 1$ ; it vanishes for  $|z| > 1$ .

The next integration to be carried out is the one over  $d^3g$ . By means of the expression (3.25) we obtain for the integral (3.20)

$$(2\pi)^2 M/2c' \int_0^\infty g dg \cdot \int_{\cos \Theta_0}^{\cos \Theta_1} d \cos \Theta P_l(\cos \Theta) P_l(z(\cos \Theta)) \cdot \exp \left\{ -\beta_j \left[ g^2 + \left( \frac{1-M}{M} \right)^2 c^2 - 2gc \frac{1-M}{M} \cos \Theta \right] \right\}. \quad (3.26)$$

In order to discuss the limits of the integration over  $d \cos \Theta$ , we invert Eq. (3.24).

$$\cos \Theta(z) = \frac{c'}{c} z + \frac{c^2 - c'^2}{2Mgc'}, \quad (3.27)$$

and obtain

$$\cos \Theta_1 = \min(1, \cos \Theta(1)) \quad (3.28)$$

as well as

$$\cos \Theta_0 = \max(-1, \cos \Theta(-1)). \quad (3.29)$$

Moreover the integral vanishes unless

$$\cos \Theta_1 > \cos \Theta_0. \quad (3.30)$$

As a consequence of a detailed analysis of the relations (3.28)–(3.30) we now have the following integration scheme:

$$I_1 \equiv \int_0^c dc' \int_{(c-c')/2M}^{(c+c')/2M} dg \int_{\cos \Theta(-1)}^1 d \cos \Theta \dots, \quad (3.31)$$

$$I_2 \equiv \int_0^c dc' \int_{(c+c')/2M}^\infty dg \int_{\cos \Theta(-1)}^{\cos \Theta(+1)} d \cos \Theta \dots, \quad (3.32)$$

$$I_3 \equiv \int_c^\infty dc' \int_{(c'-c)/2M}^{(c+c')/2M} dg \int_{-1}^{\cos \Theta(+1)} d \cos \Theta \dots, \quad (3.33)$$

and

$$I_4 \equiv \int_c^\infty dc' \int_{(c+c')/2M}^\infty dg \int_{-1}^{+1} d \cos \Theta \dots \quad (3.34)$$

The integrand is the same for all four integrals:

$$(2\pi)^2 M^{-2} \sigma_{RS} n_j (\beta_j/\pi)^{3/2} \cdot c' L_l^n(c') g P_l(\cos \Theta) P_l(z(\cos \Theta)) \cdot \exp \left\{ -\beta_j \left[ g^2 + \left( \frac{1-M}{M} \right)^2 c^2 - 2 \frac{1-M}{M} gc \cos \Theta \right] \right\}. \quad (3.35)$$

As we just want to show the essential features of our procedure, we restrict ourselves in the final discussions to the case  $l=0$  as well as  $M=1/2$ . Then the integration over  $d \cos \Theta$  is trivial, and after some elaborate but elementary manipulations we are left with

$$\sum_{k=1}^4 I_k = v_{RS}/\sqrt{\varepsilon} \left\{ \int_0^\varepsilon d\varepsilon' L_0^n(\varepsilon') \gamma(\varepsilon', \frac{1}{2}) + \int_\varepsilon^\infty d\varepsilon' L_0^n(\varepsilon') e^{-\varepsilon'} \gamma(\varepsilon, \frac{1}{2}) e^\varepsilon \right\} \quad (3.36)$$

with the notation of Eqs. (3.15), (3.16), and (3.19). Combining Eqs. (3.36), (3.17), and (3.11), we thus end up with the following integral equation for the isotropic ( $l=0$ ) part of  $L$ :

$$\sqrt{\varepsilon} \Phi_0^n(\varepsilon) = v_{RS} \left\{ k(\varepsilon) L_0^n(\varepsilon) - \int_0^\varepsilon d\varepsilon' L_0^n(\varepsilon') \gamma(\varepsilon', \frac{1}{2}) - \Gamma(\varepsilon) e^\varepsilon \int_\varepsilon^\infty d\varepsilon' L_0^n(\varepsilon') e^{-\varepsilon'} \right\}. \quad (3.37)$$

The solution of this equation is discussed in the next section. Here we add as concluding remark that Eq. (3.37) can already be used to determine the matrix elements of the inverse collision operator. They appear in the expansion

$$L_0^n(\varepsilon) = L_n^n(0) \Phi_0^{n'}(\varepsilon) \quad (3.38)$$

implying the integral representation

$$L_n^n(0) = \sqrt{2} \int_0^\infty d\varepsilon \sqrt{\varepsilon} L_0^n(\varepsilon) \Phi_{n',0}(\varepsilon). \quad (3.39)$$

However, we are able to determine the analytic form of the inverse collision kernel directly.

#### IV. Solution for $l=0$

Equation (3.37) determines the functions  $L_0^n(\varepsilon)$ . However, we can get rid of the index  $n$ , multiplying by the basis functions  $\Phi_{n,0}(\varepsilon)$  and summing up



over  $n$ . We define

$$L_0(\bar{\varepsilon}, \varepsilon) \equiv v_{RS} \sum_{n \neq 0} \Phi_{n,0}(\bar{\varepsilon}) L_0^n(\varepsilon), \quad (4.1)$$

use the restricted completeness relation (2.1) in the form

$$\begin{aligned} \sum_{n \neq 0} \Phi_0^n(\varepsilon) \Phi_{n,0}(\bar{\varepsilon}) &= (2\varepsilon)^{-1/2} \delta(\varepsilon - \bar{\varepsilon}) - (2/\pi)^{1/2} e^{-\bar{\varepsilon}} \\ &\equiv h(\bar{\varepsilon}, \varepsilon)/\sqrt{\varepsilon}, \end{aligned} \quad (4.2)$$

and obtain from Eq. (3.37) upon differentiation the simple differential equation

$$-e^{-\varepsilon} \frac{d}{d\varepsilon} h(\bar{\varepsilon}, \varepsilon) = \frac{d}{d\varepsilon} k(\varepsilon) e^{-\varepsilon} \frac{d}{d\varepsilon} I(\bar{\varepsilon}, \varepsilon) \quad (4.3)$$

for the function

$$I(\bar{\varepsilon}, \varepsilon) = e^\varepsilon \int_{\varepsilon}^{\infty} L_0(\bar{\varepsilon}, \varepsilon') e^{-\varepsilon'} d\varepsilon'. \quad (4.4)$$

(For a derivation of Eq. (4.3) note that

$$\frac{d}{d\varepsilon} k(\varepsilon) = \gamma(\varepsilon, \tfrac{1}{2}) + e^{-\varepsilon}/\sqrt{\varepsilon}.) \quad (4.5)$$

The isotropic part of the inverse collision kernel,  $L_0(\bar{\varepsilon}, \varepsilon)$ , can then be obtained as

$$L_0(\bar{\varepsilon}, \varepsilon) = I(\bar{\varepsilon}, \varepsilon) - \frac{d}{d\varepsilon} I(\bar{\varepsilon}, \varepsilon). \quad (4.6)$$

The integration of Eq. (4.3) looks trivial; the determination of the constants of integration, however, needs some discussion. As a preliminary comment we note that

$$\lim_{\varepsilon \rightarrow 0} k(\varepsilon) e^{-\varepsilon} \frac{d}{d\varepsilon} I(\bar{\varepsilon}, \varepsilon) = -\lim_{\varepsilon \rightarrow 0} k(\varepsilon) L_0(\bar{\varepsilon}, \varepsilon) = 0, \quad (4.7)$$

as a consequence of (3.37). Hence, the first integration of (4.3) yields

$$-\int_0^{\varepsilon} e^{-\varepsilon'} \frac{d}{d\varepsilon'} h(\bar{\varepsilon}, \varepsilon') d\varepsilon' = k(\varepsilon) e^{-\varepsilon} \frac{d}{d\varepsilon} I(\bar{\varepsilon}, \varepsilon). \quad (4.8)$$

Rewriting this equation, we obtain

$$\begin{aligned} \frac{d}{d\varepsilon} I(\bar{\varepsilon}, \varepsilon) &= -\frac{1}{k(\varepsilon)} e^\varepsilon \\ &\cdot \left[ e^{-\varepsilon} h(\bar{\varepsilon}, \varepsilon) + \int_0^{\varepsilon} e^{-\varepsilon'} h(\bar{\varepsilon}, \varepsilon') d\varepsilon' \right]. \end{aligned} \quad (4.9)$$

The integral

$$I_0(\bar{\varepsilon}) \equiv I(\bar{\varepsilon}, 0) = \int_0^{\infty} L_0(\bar{\varepsilon}, \varepsilon') e^{-\varepsilon'} d\varepsilon' \quad (4.10)$$

appears as boundary value for the second integration:

$$\begin{aligned} I(\bar{\varepsilon}, \varepsilon) &= I_0(\bar{\varepsilon}) - \int_0^{\varepsilon} k^{-1}(\varepsilon') h(\bar{\varepsilon}, \varepsilon') d\varepsilon' \\ &\quad - \int_0^{\varepsilon} k^{-1}(\varepsilon') e^{\varepsilon'} \int_0^{\varepsilon'} e^{-\varepsilon''} h(\bar{\varepsilon}, \varepsilon'') d\varepsilon''. \end{aligned} \quad (4.11)$$

According to Eq. (4.4) we are left with

$$\begin{aligned} L_0(\bar{\varepsilon}, \varepsilon) &= I_0(\bar{\varepsilon}) - \int_0^{\varepsilon} k^{-1}(\varepsilon') h(\bar{\varepsilon}, \varepsilon') d\varepsilon' \\ &\quad - \int_0^{\varepsilon} k^{-1}(\varepsilon') e^{\varepsilon'} \int_0^{\varepsilon'} e^{-\varepsilon''} h(\bar{\varepsilon}, \varepsilon'') d\varepsilon'' \\ &\quad + k^{-1}(\varepsilon) h(\bar{\varepsilon}, \varepsilon) + k^{-1}(\varepsilon) e^\varepsilon \\ &\quad \cdot \int_0^{\varepsilon} e^{-\varepsilon'} h(\bar{\varepsilon}, \varepsilon') d\varepsilon'. \end{aligned} \quad (4.12)$$

For a discussion of this result we check the operators  $PL^{-1}$  and  $L^{-1}P$  (cf. Equation (3.3)). Expressed in terms of the variable  $\varepsilon$  we have

$$PL^{-1} = \Phi_{0,0}(\bar{\varepsilon}) \int_0^{\infty} d\bar{\varepsilon}' (2\bar{\varepsilon}')^{1/2} L_0(\bar{\varepsilon}', \varepsilon). \quad (4.13)$$

By means of Eq. (4.12) this relation yields

$$PL^{-1} = \Phi_{0,0}(\bar{\varepsilon}) \int_0^{\infty} d\bar{\varepsilon}' (2\bar{\varepsilon}')^{1/2} I_0(\bar{\varepsilon}'), \quad (4.14)$$

as we have (cf. Eq. (4.2))

$$\int_0^{\infty} \sqrt{\bar{\varepsilon}} h(\bar{\varepsilon}, \varepsilon) d\bar{\varepsilon} = 0. \quad (4.15)$$

Equation (4.14) poses a first condition to the integral  $I_0(\bar{\varepsilon})$ . On the other hand, requiring that the operator  $L^{-1}P$  vanishes as well, we fix the integral  $I_0(\bar{\varepsilon})$  as demonstrated in the following equation

$$\begin{aligned} I_0(\bar{\varepsilon}) \int_0^{\infty} \sqrt{\varepsilon} e^{-\varepsilon} d\varepsilon \\ = \int_0^{\infty} \sqrt{\varepsilon} e^{-\varepsilon} \left[ F(\varepsilon) - \int_0^{\varepsilon} F(\varepsilon') d\varepsilon' \right] d\varepsilon. \end{aligned} \quad (4.16)$$

Here we have used the abbreviation

$$\begin{aligned} F(\varepsilon) &\equiv k^{-1}(\varepsilon) \left[ h(\bar{\varepsilon}, \varepsilon) + e^\varepsilon \int_0^{\varepsilon} e^{-\varepsilon'} h(\bar{\varepsilon}, \varepsilon') d\varepsilon' \right] \\ &= (\sqrt{2} k(\varepsilon))^{-1} [\delta(\varepsilon - \bar{\varepsilon}) - 2(\varepsilon/\pi)^{1/2} e^{-\bar{\varepsilon}} \\ &\quad + e^{\varepsilon - \bar{\varepsilon}} (\Theta(\varepsilon - \bar{\varepsilon}) - 1)]. \end{aligned} \quad (4.17)$$

By means of partial integration we obtain instead of (4.16)

$$I_0(\bar{\varepsilon}) = \sqrt{\pi}^{-1} \int_0^{\infty} (\gamma(\varepsilon, \tfrac{1}{2}) - \sqrt{\pi}) F(\varepsilon) d\varepsilon, \quad (4.18)$$

or explicitly

$$I_0(\bar{\varepsilon}) = (2\pi)^{-1/2} \left[ (\gamma(\bar{\varepsilon}, \tfrac{1}{2}) - \sqrt{\pi})/k(\bar{\varepsilon}) - \frac{2}{\sqrt{\pi}} \int_0^{\bar{\varepsilon}} \sqrt{\varepsilon} \frac{\gamma(\varepsilon, \tfrac{1}{2}) - \sqrt{\pi}}{k(\varepsilon)} d\varepsilon - e^{-\bar{\varepsilon}} \int_0^{\bar{\varepsilon}} e^{\varepsilon} \frac{\gamma(\varepsilon, \tfrac{1}{2}) - \sqrt{\pi}}{k(\varepsilon)} d\varepsilon \right]. \quad (4.19)$$

The infinite integral in this equation converges because of

$$\sqrt{\pi} - \Gamma(\varepsilon) \rightarrow e^{-\varepsilon}/\sqrt{\varepsilon} \quad (\varepsilon \rightarrow \infty) \quad (4.20)$$

which can be shown by means of De l'Hospital's rule:

$$\lim_{\varepsilon \rightarrow \infty} \frac{\sqrt{\pi} - \gamma(\varepsilon, \tfrac{1}{2})}{e^{-\varepsilon}/\sqrt{\varepsilon}} = \lim_{\varepsilon \rightarrow \infty} \frac{\varepsilon}{\varepsilon + 1/2} = 1. \quad (4.21)$$

We conclude the discussions of this section with a remark concerning the direct calculation of the matrix elements of the inverse collision operator. Basic considerations of this section are valid for arbitrary inhomogeneities  $h(\bar{\varepsilon}, \varepsilon)$ , especially for the original one appearing in (3.37). Therefore, corresponding to Eq. (4.2), let

$$h^n(\varepsilon) \equiv \sqrt{\varepsilon} \Phi_0^n(\varepsilon) \quad (4.22)$$

and, in correspondence to the definition (4.17),

$$F^n(\varepsilon) \equiv k^{-1}(\varepsilon) \left[ \sqrt{\varepsilon} \Phi_0^n(\varepsilon) + e^{\varepsilon} \int_0^{\varepsilon} e^{-\varepsilon'} \sqrt{\varepsilon'} \Phi_0^n(\varepsilon') d\varepsilon' \right]. \quad (4.23)$$

Then the solution of Eq. (3.37) can be expressed immediately as (cf. (4.12))

$$\nu_{RS} \mathcal{L}_0^n(\varepsilon) = I_0^n(0) + F^n(\varepsilon) - \int_0^{\varepsilon} F^n(\varepsilon') d\varepsilon' \quad (4.24)$$

with the constant  $I_0^n(0)$ . Finally, carrying out the integration (3.39), we obtain

$$\nu_{RS} \mathcal{L}_{n'}^n(0) = \int_0^{\infty} d\varepsilon (2\varepsilon)^{1/2} \Phi_{n',0}(\varepsilon) \cdot \left[ F^n(\varepsilon) - \int_0^{\varepsilon} F^n(\varepsilon') d\varepsilon' \right]. \quad (4.25)$$

Here we can see explicitly that because of the orthogonality of the basis functions, the choice of

the constant  $I_0^n(0)$  has no influence on the matrix elements at all. The evaluation of Eq. (4.25) is a matter of straight-forward application of the algebraic properties of the basis system and shall not be discussed further.

## V. Concluding Remarks

In the foregoing sections we have discussed an approach to the determination of the analytic form of the inverse collision operator. Apart from standard physical restrictions such as conservation of total momentum, energy, and relative angular momentum during the (binary) collisions, a couple of further assumptions have been made. The first and probably the most essential one has been the requirement that the interaction potential has a cut-off structure. Only then the collision operator can be separated into two terms, one of them being purely multiplicative, a procedure that is needed for a vast variety of mathematical treatments concerning the spectral properties of the operators.

Less serious assumptions are the restrictions to rigid sphere interactions, to the isotropic part with  $l=0$ , and to equal masses of the collision partners. They merely minimize the analytical efforts and can be overcome by a more elaborate evaluation of our expression (3.35). Such a generalization as well as a discussion of collision operators for potentials with no cut-off structure are to be reserved for subsequent papers.

## Appendix

### Burnett Basis Functions

Let  $\mathbf{x}$  be the vector argument of our basis functions  $\Phi_{n,l,m}$ . Its physical dimension shall be taken care of by the normalizing constant  $\gamma$  appearing in the definition

$$\varepsilon \equiv \frac{1}{2} \left( \frac{x}{\gamma} \right)^2. \quad (A1)$$

As the indices  $l, m$  are referred to the spherical harmonics  $Y_{l,m}(\vartheta, \varphi)$ ,  $(\vartheta, \varphi)$  being the solid angle of the vector  $\mathbf{x}$ , we define

$$\Phi_{n,l,m}(\mathbf{x}) \equiv \Phi_{n,l}(\varepsilon) \gamma^{-(n+3)} Y_{l,m}(\vartheta, \varphi). \quad (A2)$$

The "radial" part  $\Phi_{n,l}(\varepsilon)$  and the power of  $\gamma$  can be derived by the following requirements:

a) For  $n=0$  the basis function is to be the equilibrium distribution function, apart from the particle density,

$$\Phi_{0,0,0}(\mathbf{x}) \equiv (2\pi\gamma^2)^{-3/2} e^{-\varepsilon}. \quad (\text{A3})$$

b) The dual basis functions

$$\Phi_{n,l,m}^{n,l,m}(\mathbf{x}) \equiv \Phi_l^n(\varepsilon) \gamma^n Y_{l,m}^*(\vartheta, \varphi) \quad (\text{A4})$$

defined by the duality relation

$$\int d^3x \Phi_{n',l',m'}^{n',l',m'}(\mathbf{x}) \Phi_{n,l,m}(\mathbf{x}) = \delta_{n'}^{n'} \delta_{l'}^{l'} \delta_{m'}^{m'}, \quad (\text{A5})$$

as well as the basis functions themselves, can be looked upon as spherical components [24] of an  $n$ -th order symmetric tensor of rank  $l$ , with  $l \leq n$  and  $n-l$  even. Especially for  $l=n$ , i.e. for the maximum rank, we require that the dual basis functions  $\Phi_{n,n,m}$  are the spherical components of the completely symmetric and traceless (irreducible)  $n$ -th order tensorial power of  $\mathbf{x}$ .

Both requirements are satisfied by the combination of

$$\Phi_{n,l}(\varepsilon) = (-1)^{\frac{1}{2}(n-l)} 2^{\frac{1}{2}(l+1)} \cdot \frac{(2l+1)^{1/2}}{2\pi} \frac{1}{(n+l+1)!!} \varepsilon^{l/2} e^{-\varepsilon} L_{\frac{1}{2}(n-l)}^{(l+\frac{1}{2})}(\varepsilon) \quad (\text{A6})$$

with

$$\Phi_l^n(\varepsilon) = (-1)^{\frac{1}{2}(n-l)} 2^{\frac{1}{2}(l+1)} \cdot \left( \frac{2\pi}{2l+1} \right)^{1/2} (n-l)!! \varepsilon^{l/2} L_{\frac{1}{2}(n-l)}^{(l+\frac{1}{2})}(\varepsilon). \quad (\text{A7})$$

A basic relation for the determination of the normalizing coefficients in Eqs. (A6) and (A7) is the orthogonality relation for the Laguerre polynomials,

$$\int_0^\infty d\varepsilon \varepsilon^{l+\frac{1}{2}} L_{\frac{1}{2}(n'-l)}^{(l+\frac{1}{2})}(\varepsilon) L_{\frac{1}{2}(n-l)}^{(l+\frac{1}{2})}(\varepsilon) e^{-\varepsilon} = \delta_{n'}^{n'} 2^{-(l+1)} \frac{(n+l+1)!!}{(n-l)!!} \sqrt{\pi}. \quad (\text{A8})$$

The corresponding completeness relation

$$\sum_n 2^{l+1} \frac{(n-l)!!}{(n+l+1)!!} \frac{1}{\sqrt{\pi}} \cdot \varepsilon'^{l/2} \varepsilon^{l/2} L_{\frac{1}{2}(n-l)}^{(l+\frac{1}{2})}(\varepsilon') L_{\frac{1}{2}(n-l)}^{(l+\frac{1}{2})}(\varepsilon) \cdot e^{-\varepsilon} \sqrt{\varepsilon} = \delta(\varepsilon - \varepsilon') \quad (\text{A9})$$

is what has been referred to as “restricted completeness relation” in the main part of this paper.

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